Analytic continuation of $\zeta(s)$ and $L(s,\chi)$

Recall that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ defined for $\mathrm{Re}(s) > 1$

and ζ is holomorphic on $\mathrm{Re}(s) > 1$. We can extend definition of $\zeta(s)$ to larger region.

<u>Theorem</u>: ζ admits unique meromorphic continuation to $\{s : \mathrm{Re}(s) > 0\}$ with a simple pole at $s = 1$. Moreover, $\operatorname{Res}_{s=1} \zeta(s) = 1$.

<u>Proof</u>: Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. By Abel summation,

$$\sum_{n \le x} \frac{1}{n^s} = \sum_{\frac{1}{2} < n \le x} \frac{\mathbb{1}(n)}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_{\frac{1}{2}}^{x} \frac{\lfloor t \rfloor}{t^{s+1}}\, dt$$

$$= \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{(t - \{t\})}{t^{s+1}}\, dt$$

$$= \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{1}{t^s} - s \int_1^x \frac{\{t\}}{t^{s+1}}\, dt$$

integral converges as $x \to \infty$

$$= \overset{x\to\infty}{\longrightarrow 0}\frac{\lfloor x \rfloor}{x^s} + \frac{s}{s-1} - \overset{x\to\infty}{\longrightarrow 0}\frac{x^{1-s}}{s-1} - s \int_1^x \frac{\{t\}}{t^{s+1}}\, dt.$$

We let $X \to \infty$, thus we obtain for $\mathrm{Re}(s) > 1$

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{\{t\}}{t^{s+1}}\, dt. \quad (*)$$

$\to$ the function $1 + \frac{1}{s-1}$ is meromorphic in entire complex plane, simple pole at $s=1$, and residue 1.

$\to$ for $\mathrm{Re}(s) > 0$, the integral converges absolutely. Thus defines holomorphic function on $\mathrm{Re}(s) > 0$.

Hence, RHS of $(*)$ defines analytic function on $\mathrm{Re}(s) > 0$ with simple pole at $s=1$, which coincides with $\sum_{n=1}^\infty n^{-s}$ on $\mathrm{Re}(s) > 1$.

The holomorphic extension of $\zeta(s)$ to $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\} - \{1\}$ is unique, as it completely determined on $\{\mathrm{Re}(s) > 1\}$.

(B)

Theorem (Analytic continuation for $L(s;\chi_0)$).

Let χ_0 be the principal character mod q. Then $L(s,\chi_0)$ admits unique meromorphic continuation to $\mathrm{Re}(s) > 0$, with a simple pole at $s=1$

and residue $\operatorname{Res}_{s=1} L(s,\chi_0) = \frac{\varphi(q)}{q}$.

Proof: By Euler product, for $\operatorname{Re}(s) > 1$,

$$L(s,\chi_0) = \prod_p \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1} = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$= \zeta(s) \cdot \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$

Note that $\prod_{p \mid q} \left(1 - \frac{1}{p^s}\right)$ defines a holomorphic function for all $s \in \mathbb{C}$ and that $\prod_{p \mid q} \left(1 - \frac{1}{p}\right) = \frac{\varphi(q)}{q}$.

Conclusion follows from meromorphic continuation of $\zeta(s)$ to $\operatorname{Re}(s) > 0$. $\square$

Theorem (Analytic continuation for non-principal characters).

If χ mod q NOT the principal character, then $\sigma_c(\chi) = 0$. In particular, $L(s,\chi)$ has holomorphic continuation to $\operatorname{Re}(s) > 0$.

(so no poles)

Proof: Note that $\sum_{y < n \leq y+q} \chi(n) = 0$, $\forall y > 0$.

Define $S_\chi(N) := \sum_{n \leq X} \chi(n)$. Then $|S_\chi(N)| \leq \varphi(q)$.

We need to show $\left| \sum_{y < n \leq X} \frac{\chi(n)}{n^s} \right| < \varepsilon$, whenever

$M_\varepsilon < y < X$ and $\mathrm{Re}(s) > 0$.

By Abel summation,

$$\sum_{y < n \leq X} \frac{\chi(n)}{n^s} = \frac{S_\chi(X)}{X^s} - \frac{S_\chi(y)}{y^s} + s \int_y^X \frac{S_\chi(t)}{t^{s+1}} dt$$

Hence $$\left| \sum_{y < n \leq X} \frac{\chi(n)}{n^s} \right| \ll \varphi(q) \left(\frac{1}{X^\sigma} + \frac{1}{y^\sigma} \right) + |s| \varphi(q) \int_y^X \frac{1}{t^{\sigma+1}} dt$$

$$= \varphi(q)(X^{-\sigma} + y^{-\sigma}) + \frac{|s| \varphi(q)}{\sigma} (y^{-\sigma} - X^{-\sigma}) \longrightarrow 0$$

as $Y \to \infty$.

(if $\sigma > 0$).

We note $\sum_{n \leq X} \frac{\chi(n)}{n^s}$ convergent for $\mathrm{Re}(s) > 0$

$\Rightarrow \sigma_c(\chi) = 0$

$\Rightarrow L(s, \chi)$ holomorphic for $\mathrm{Re}(s) > 0$. $\blacksquare$

Theorem (Dirichlet's theorem on primes in arithmetic progressions)

Let q, a coprime integers.

Then $\{ p \in \mathbb{P} : p \equiv a \bmod q \}$ is infinite.

Motivation for proof: There are infinitely many primes.

For $\sigma > 1$, we have $\zeta(\sigma) = \prod_p \left(1 - \frac{1}{p^\sigma}\right)^{-1}$

$$\Rightarrow \log \zeta(\sigma) = -\sum_p \log\left(1 - \frac{1}{p^\sigma}\right) = \sum_p \frac{1}{p^\sigma} + O(1)$$

(use Taylor expansion $\log(1-y)$)

But we know $\lim_{\sigma \downarrow 1} \zeta(\sigma) = \infty$

$$\Rightarrow \lim_{\sigma \downarrow 1} \sum_p \frac{1}{p^\sigma} = \infty$$

$$\Rightarrow \sum_p \frac{1}{p} = \infty \Rightarrow \text{there are infinitely many primes.}$$

(proof is direct consequence of Euler product).

Proof preparation for Dirichlet's theorem:

We will show $\sum_{p \equiv a (q)} \frac{1}{p}$ is infinite.

By orthogonality relations,

$$\sum_{\substack{p \le x \\ p \equiv a (q)}} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \underbrace{\sum_{p \le x} \frac{\chi(p)}{p^s}}_{=: R_\chi(x)}$$

Recall for $Re(s) > 1$, $L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$.

$$\Rightarrow \log(L(s,\chi)) = -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) + 2\pi i m,$$

for some $m \in \mathbb{Z}$.

Let $s \to \infty$ (along $\mathbb{R}$), we have $L(s,\chi) \to 1$, and similarly for RHS

$\Rightarrow m = 0$.

$$\log(L(s,\chi)) = -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) = \underbrace{\sum_p \frac{\chi(p)}{p^s}}_{= \lim_{x \to \infty} R_\chi(x)} + O(1)$$

For $Re(s) > 1$,

$$\Rightarrow \sum_{p \equiv a (q)} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \log L(s,\chi) + O(1). \quad (*)$$

→ need to make sense of $L(s,\chi)$ at $s = 1$.

Want to take limit as $\sigma \searrow 1$.
We know $\lim_{\sigma \downarrow 1} \log(L(\sigma, \chi_0)) = \infty$ and want

to show the other contribution are finite. Hence Dirichlet's theorem follows from:

Theorem: (Dirichlet) Let $\chi \neq \chi_0$. Then $L(1, \chi) \neq 0$.

Corollary: $\lim_{\sigma \downarrow 1} \sum_{p \equiv a(q)} \frac{1}{p^\sigma} = \infty$.

Proof: We note that for $\sigma > 1$,

$$\sum_{\chi \bmod q} \log(L(\sigma, \chi)) = \sum_{\chi \bmod q} \sum_{p} \sum_{l=1}^{\infty} \frac{\chi(p)^l}{l p^{l\sigma}}$$

$$= \sum_{p} \sum_{l=1}^{\infty} \frac{1}{l p^{l\sigma}} \sum_{\chi \bmod q} \chi(p^l) \quad \text{(from absolute convergence)}$$

$$= \sum_{p} \sum_{l : p^l \equiv 1(q)} \varphi(q) \frac{1}{l p^{l\sigma}} > 0.$$

$\Rightarrow$ for $\sigma > 1$, $\prod_{\chi \bmod q} L(\sigma, \chi) > 1$.

Recall for $\chi \neq \chi_0$, $L(s,\chi)$ holomorphic on $\mathrm{Re}(s)>0$,

so $L(s,\chi) = (s-1)^{h_\chi} \cdot M_\chi(s)$,

where $h_\chi \in \mathbb{N} \cup \{0\}$ (order of the zero at 1).

$M_\chi(s)$ holomorphic on $\mathrm{Re}(s)>0$, $M_\chi(1) \neq 0$.

For χ_0, $L(s,\chi_0)$ has simple pole at $s=1$.

$$\Rightarrow \prod_{\chi \bmod q} L(s,\chi) = M(s) \cdot (s-1)^{-1+\sum_{\chi\neq\chi_0} h_\chi},$$

where M holomorphic on $\mathrm{Re}(s)>0$.

Since $\{\prod_{\chi \bmod q} L(\sigma,\chi) : \sigma > 1\}$ is bounded away from 0,

we must have $\sum_{\chi\neq\chi_0} h_\chi \leq 1$

(there is at most one $\chi \bmod q$ s.t. $L(1,\chi)=0$).

<u>Step 1</u>: Non-vanishing for complex characters.

We show $h_\chi = h_{\overline{\chi}}$.

Indeed, $L(s,\overline{\chi}) = \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^s} = \overline{\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\overline{s}}}}$ $(\overline{n^s} = n^{\overline{s}})$

$= \overline{L(\overline{s},\chi)}$ for $\mathrm{Re}(s)>1$.

From meromorphic continuation, identity holds also for $\mathrm{Re}(s)>0$.

Hence $L(1,\chi)=0 \Rightarrow L(1,\overline{\chi})=0$.
So if $\chi\neq\overline{\chi}$, we must have $h_\chi = h_{\overline{\chi}} = 0$.

Step 2: Non-vanishing for non-principal real (quadratic) characters.

Suppose $\chi=\overline{\chi}$, i.e. $\chi^2=\chi_0$ ($\chi(n)\in\{0,\pm 1\}$)
but $\chi\neq\chi_0$.

Define $r(n)=\sum_{d|n}\chi(d) = \chi * \varepsilon(n)$
$\rightarrow r$ multiplicative

We have that $r(p^l) = \begin{cases} 1, & \text{if } p|q \\ l+1, & \text{if } (p,q)=1 \text{ and } \chi(p)=1 \\ 1, & \text{if } \chi(p)=-1 \text{ and } l \text{ even} \\ 0, & \text{if } \chi(p)=-1 \text{ and } l \text{ odd} \end{cases}$

(exercise)

In particular, $r(n)\geq 0$ and $r(n^2)\geq 1$, $\forall n\in\mathbb{N}$.

Suppose for contradiction $L(1,\chi)=0$.
Then $L_r(s)=L_\chi(s)\zeta(s)$ holomorphic for $\{\operatorname{Re} s>0\}$.

[since $\zeta(s) = \frac{1}{s-1} M_\zeta(s)$, $M_\zeta(s)$ & $M_\chi(s)$ holomorphic on $\Re(s)>0$

$L_\chi(s) = (s-1)^{h_\chi} M_\chi(s)$

$\Rightarrow L_r(s) = (s-1)^{h_\chi - 1} M_\chi(s) M_\zeta(s)$]

Theorem (Landau)

Let $f \in \mathcal{A}$ with $\sigma_c(f) < \infty$ and $f(n) \geq 0$, $\forall n \in \mathbb{N}$. Then $L_f(s)$ has singularity (i.e. not holomorphic) at $s = \sigma_c(f)$.

Assuming Landau, it follows $\sigma_c(r) \leq 0$.

$$\text{But } \sum_{n \leq X} \frac{r(n)}{\sqrt{n}} \underset{r(n)\geq 0}{\geq} \sum_{n \leq \sqrt{X}} \frac{r(n^2)}{n} \underset{r(n^2)\geq 1}{\geq} \sum_{n \leq \sqrt{X}} \frac{1}{n} = \frac{1}{2} \log x + O(1)$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{r(n)}{n^{1/2}}$ diverges, contradiction.

This concludes proof of Dirichlet's theorem. $\square$

Remark: From the proof, it follows that if χ non-principal quadratic character and $\sigma > 1$, $\zeta(\sigma) L_\chi(\sigma) = \sum_n \frac{r(n)}{n^\sigma} > 0$.

As $\zeta(\sigma) > 0 \Rightarrow L(1, \chi) > 0$.